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A new analytical formulation for investigating in modern engineering for the harmonic distortion occurring at large vibration amplitudes of clamped-clamped beams: Explicit Solutions

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ABSTRACT: This work is a contribution to the numerical modeling and computer implementation of geometrically non-linear vibrations of the thin beams. The spatial distribution over the beam span of the harmonic distortion induced by large vibration amplitudes has been examined, and an analytical investigation has been elaborated to describe this aspect of non-linear vibration. This model allowed us to obtain the explicit analytical expressions for the non-linear response, including the contributions of various spatial functions, associated to the first and higher time harmonics. In the present work, devoted to this particular but practically very important aspect of non-linear vibration, a review is made of some important experimental and theoretical works on the subject. The model is based on an expansion of the transverse displacement function as a sum of series, each series being the product of a given harmonic time function by a series of chosen basic spatial functions, multiplied by the unknown contribution coefficients, to be determined. The explicit analytical expressions obtained for the function contributions corresponding to the first time harmonic are identical to those obtained in the previous works above assuming harmonic motion, which allows one to consider that the present model is a generalization of the previous ones. Also, the results of the model presented here, corresponding to the higher harmonics, are in a very close agreement with each other. They are also in a qualitative agreement with previously published numerical results, based on the hierarchical finite element method.

KEYWORDS: Vibration, non-linear, harmonic distortion, amplitude, multimode.

1 INTRODUCTION

The previous work was concerned with the non linear response of clamped-clamped beams to harmonic excitation exhibits, at large vibration amplitudes. In modern engineering problems, large vibration amplitudes of beam-like or plate-like structures very often occur [1], inducing a dynamic behaviour, which is different in many ways from that predicted by linear structural dynamics theories.

A theoretical model based on Hamilton's principle and spectral analysis has been developed and used to study the non-linear free and steady state periodic forced vibration of beams, homogeneous and composite plates [2], [3], [4], [5], [6], [7], [8], [9], [10] [11], [12]. The model developed in [9] has been extended to the non-linear free vibration of cylindrical shells. The effects of large vibration amplitudes on the first and second coupled transverse-circumferential mode shapes of isotropic circular cylindrical shells of infinite length have been examined in [13], [14],[15]. This model was used to calculate the second non-linear mode of fully clamped homogeneous rectangular plates with various values of the aspect ratio, and to analyse the effect of non-linearity on the induced bending stresses. In [16], the model presented in [2], [3], [4] was adapted to study the non-linear steady state forced periodic response of C-C and S-S beams; the results obtained were close to those obtained by other methods. More recently, this method has been extended to free vibrations of clamped circular plates and C-C-C-SS plates [17], [18]. Good agreement has been found in each case via comparisons with previous published works. In reference [19], the geometrically non-linear free vibration of symmetrically laminated rectangular plates with the fully clamped boundary conditions has been examined both experimentally and theoretically. The model was validated by comparison with experimental results.

In an improved version of the model, the spatial distribution of the harmonic distortion was also introduced in the analytical and numerical formulation and some results were obtained, which are presented in [20]. In [21], a practical simple "multi-mode theory " based on the linearization of the non-linear algebraic equations, written on the modal basis, in the neighbourhood of each resonance has been developed. Simple formulae have been derived, which is easy to use, for engineering purposes. The approach has been successfully used in the free vibration case to the first, second and third non-linear mode shapes of CC beams, and the first non-linear mode shape of clamped SS beam. It has also been applied to obtain the non-linear steady state periodic forced response of CC and clamped SS beams, excited harmonically with concentrated and distributed forces.

In reference [22], the fundamental mode shape of a clamped-clamped beam has been investigated experimentally at large vibration amplitudes, and least square polynomial approximation fitting procedures were used to numerically estimate its amplitude dependence, and determine its first and second spatial derivatives. Although the excitation was harmonic, the non-linear response of the beam exhibited a harmonic distortion, which was more accentuated near to the clamps, as shown in figures 6 to 8 in [22]. A careful examination of this effect was made via separation of harmonics performed using a B&K analyser. The spatial distribution of the third harmonic component was found to be quite similar to the second mode shape of the clamped-clamped beam, which should exist theoretically at three times the fundamental frequency, according to the linear analysis. This was attributed to a probable coincidence of the third harmonic, due to the non-linear effect, with the second resonance frequency of a clamped-clamped beam. It was also concluded, on the basis of experimental measurements, that the harmonic distortion of the induced strains in the clamped-clamped beam tested are influenced by two factors: non linear effects due to large amplitude oscillations, which change the mode shape, and the contribution of the axial strain to the resultant induced strain, which is dependent upon position along the beam. A laborious theoretical analytical model has been then constructed in an attempt to obtain a more detailed explanation for the behavior of the induced total strains observed in the experimental work. In [8], the amplitude dependant first three non-linear mode shapes of clamped-clamped beams, but the harmonic distortion effect was neglected.

In reference [10], experimental and theoretical investigations of the harmonic distortion of the induced strain components in clamped-clamped beams at large deflections were carried out. It was shown that the bending strain exhibited a significant harmonic distortion due to the considerable contribution of higher harmonics, such as two and three times the excitation frequency, in addition to the fundamental component. The axial strain signal showed much less harmonic distortion, and was basically composed of a component having twice the excitation frequency, with amplitude much larger than the other harmonics. This work showed also that the non-linear vibration due to large deflection amplitudes of a clamped-clamped beam significantly affected the fatigue life, due to the influence of the in-plane strain component. The harmonic distortion had a significant effect in reducing fatigue life than steady vibration at amplitudes of the order of the beam thickness. The experimental results for the particular beam tested showed a 60% reduction in fatigue life at a strain level double the endurance limit, and 75% reduction at three times the endurance limit

Analytical formulations and experimental studies have been concerned with the effects of large vibration amplitudes on the mode shapes and resonance frequencies of a CC beams [1], [23], [24], [25]. Experimental measurements [22], showed that the dynamic response of a clamped-clamped beam, at large deflections, exhibits a spatial dependence of harmonic distortion in the response. Experimental work, reported in [22], has shown that the response harmonic distortion, occurring at large vibration amplitudes, can have a significant influence on the fatigue life of structures undergoing large amplitudes of vibration [24], [25].

The objective of this study, being an extension of the model presented in [20], is to propose a new model for non-linear vibration of CC beams taking into account the harmonic distortion, which occurs at large vibration amplitudes. In the present work, a new approach is presented, based on appropriate simplifications, and allowing direct calculation of the harmonic distortion, via solution of a set of linearised differential equations.

2 FORMULATION OF NON-LINEAR FREE VIBRATION OF CC BEAMS

Consider transverse vibrations of an isotropic beam having an area of cross section S, a length L, a thickness H, a young's modulus E, a mass per unit length ρ and a Poisson's ratio ν. The total beam strain energy can be written as the sum of the strain energy due to bending denoted as V_b, plus the axial strain energy due to the axial load induced by large deflection V_a. Vb, Va and the kinetic energy T are given by [2], [13]:

$$V_b = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 W}{\partial x^2}\right)^2 dx \quad V_a = \frac{ES}{8L}\left(\int_0^L\left(\frac{\partial W}{\partial x}\right)^2\right)^2 \quad T = \frac{1}{2}\rho S\int_0^L\left(\frac{\partial W}{\partial t}\right)^2 dx \tag{1-3}$$

in which W is the beam transverse displacement.

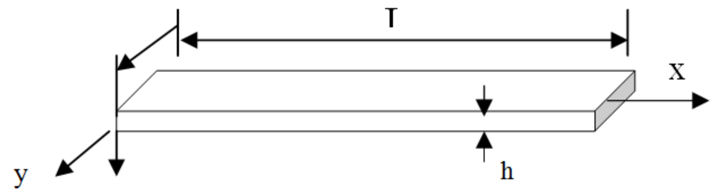

Fig. 1. Co-ordinates system for the beam

Using a generalized parameterization and the usual summation convention used in [9], the transverse displacement can be written as:

$$W(x,t) = q_i(t)w_i(x) \tag{4}$$

Substituting W in the expressions for V_b, V_a, T and rearranging leads to:

$$V_b = \frac{1}{2}q_i q_j k_{ij} \quad V_a = \frac{1}{2}q_i q_j q_k q_l b_{ijkl} \quad T = \frac{1}{2}\omega^2 \dot{q}_i \dot{q}_j m_{ij} \tag{5-7}$$

Where m_{ij}, k_{ij} and b_{ijkl} are defined in references [22]-[23] as:

$$m_{ij} = \rho S\int_0^L w_i(x)\,w_j(x)\,dx \; ; \; k_{ij} = \int_0^L\left(\frac{\partial^2 w_i}{\partial x^2}\right)\left(\frac{\partial^2 w_j}{\partial x^2}\right)dx \tag{8-9}$$

$$b_{ijkl} = \frac{ES}{4L}\int_0^L(\frac{\partial w_i}{\partial x})(\frac{\partial w_j}{\partial x})dx\int_0^L(\frac{\partial w_k}{\partial x})(\frac{w_l}{\partial x})dx \tag{10}$$

The dynamic behaviour of the structure may be obtained by Lagrange's equations for a conservative system, which leads to:

$$-\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial T}{\partial q_r} - \frac{\partial V}{\partial q_r} = 0 \quad \text{r=1 to n} \tag{11}$$

Replacing in this equation T and V =(V_a +V_b) by their expressions given above, leads to the following set of non-linear differential equations:

$$\ddot{q}_i m_{ir} + q_i k_{ir} + 2q_i q_j q_k b_{ijkr} = 0 \quad \text{r=1,..., n} \tag{12}$$

Which can be written in matrix form as:

$$[M]\{\ddot{q}\} + [K]\{q\} + 2\left[B(\{q\})\right]\{q\} = \{0\} \tag{13}$$

Where [M], [K], [B] and {q} are respectively the mass matrix, the linear rigidity matrix, the non-linear rigidity matrix depending on {q} and the column vector of generalised parameters {q}T=[q_1 q_2 … q_n].

Now assuming a harmonic motion:

$$q_i(t)=a_i \sin(\omega t) \tag{14}$$

Substituting (14) in (13) and applying the harmonic balance method leads to:

$$2\left(\left[K\right]-\omega^2\left[M\right]\right)\{A\}+3\left[B\left(A\right)\right]\{A\}=\{0\} \tag{15}$$

In which {A} is the column vector of the basic functions contribution coefficients $\{A\}^T=\{a_1\ a_2 \ldots a_n\}$.

To obtain non-dimensional parameters, we put, as in reference [22]:

$$w_i(x)=Hw_i^*\left(\frac{x}{L}\right)=Hw_i^*(x^*) \tag{16}$$

$$\frac{\omega^2}{\omega^{*^2}}=\frac{EI}{\rho SL^4} \quad \frac{k_{ij}}{k_{ij}^*}=\frac{EIH^2}{L^3} \quad \frac{m_{ij}}{m_{ij}^*}=\rho SH^2L \quad \frac{b_{ijkl}}{b_{ijkl}^*}=\frac{EIH^2}{L^3} \tag{17-20}$$

Substituting these equations in equation (15) leads to:

$$\left(\left[K^*\right]-\omega^{*^2}\left[M^*\right]\right)\{A\}+\frac{3}{2}\left[B^*\left(A\right)\right]\{A\}=\{0\} \tag{21}$$

Which may be written also, using the tensor notation as :

$$-\omega^{*^2}a_i m_{ir}^*+a_i k_{ir}^*+\frac{3}{2}a_i a_j a_k b_{ijkr}^*=0 \qquad \text{r=1 to n} \tag{22}$$

Equation (22) is identical to that obtained in reference [9] for the non-linear free vibrations of beams and plates using Hamilton's principle and integration over the range $[0, 2\pi/\omega]$. These equations are a set of non-linear algebraic equations, involving the parameters m_{ij}^*, k_{ij}^* and b_{ijkl}^* which have been computed numerically. In order to obtain the numerical solution for the non-linear problem in the neighbourhood of a given mode, the contribution of this mode is chosen and those of the other modes are calculated numerically using the Harwell library routine NS01A. For the first mode, the procedure consisted on fixing a1 and calculating the higher mode contributions from the system:

$$-\omega^{*^2}a_i m_{ir}^*+a_i k_{ir}^*+\frac{3}{2}a_i a_j a_k b_{ijkr}^*=0 \qquad \text{r>1} \tag{23}$$

in which ω^{*2} is obtained from the principle of conservation of energy as :

$$\omega^{*^2}=\frac{a_i a_j k_{ij}^*+a_i a_j a_k a_l b_{ijkl}^*}{a_i a_j m_{ij}^*} \tag{24}$$

3 A NEW SIMPLIFIED MODEL FOR BEAMS NON-LINEAR VIBRATION TAKING INTO ACCOUNT THE HARMONIC DISTORTION

3.1 GENERAL PRESENTATION

The purpose of this subsection is to replace the solution process described in the above subsection, based on the harmonic motion assumption, use of the harmonic balance method, and the iterative numerical solution of the resulting set of non-linear algebraic equations, necessary to obtain the beam non-linear mode shapes and resonance frequencies at large vibration amplitudes, by an analytical model, using the assumption, based on experimental data and a recent theoretical work [2-4], that the first time function is predominant, compared to the higher order time functions i.e, $\left|q_1(t)\right| \geq \left|q_i(t)\right|$ for i=2 to n. Then, assuming an expression for the first time function involving the first and third harmonics, which is conform to the known data, the resulting system of differential equations in time is analytically solved by superposition to obtain directly the higher order time functions. As stated in the introduction, this formulation has been developed with two objectives: 1) to develop a new approximate but efficient analytical method of solution of the multidimensional Duffing equation, which may be applicable to various non-linear systems. 2) To validate the numerical result of the first model. 3) To provide designers, needing accurate estimates of the structural fatigue life, with explicit analytical expressions, which may be easily implemented in their models, for the non-linear harmonically distorted response.

To obtain in [14] explicit solutions for the non-linear algebraic system (15), it was noticed that the results obtained iteratively in [13] allowed one to write the contribution vector $\{q\}^T$ as: $\{q\}^T = \sin(\omega t)\ [a_1\ \varepsilon_2 \ldots \varepsilon_n]$, with $\varepsilon i << a_1$, for i >1. Then, both first and second order terms with respect to εi were neglected in the term $a_i\, a_j\, a_k\, b^*_{ijkl}$ of equation (22), which has led to explicit expressions for εi, for i >1.

In the present work, a simple solution of the harmonic distortion problem is attempted, based on the assumption of an unknown, time dependent, contribution vector $\{q(t)\}^T = [q_1\ \zeta(t)_2 \ldots \zeta(t)_n]$, in which the time functions ζi's are assumed to be small, compared with $q^1(t)$. This allows, after simplifying equation (12), in a manner similar to that adopted in [14], to write the non-linear differential system (12) as:

$$\ddot{\zeta}_r(t) m^*_{rr} + \zeta_r k^*_{rr} + 2\zeta_1^3(t) b^*_{111r} = 0 \quad \text{for } r=1,\ldots,n \tag{25}$$

So that if the solution for q_1 obtained from the first equation, corresponding to r = 1, is assumed, the remaining equations, i.e.:

$$\ddot{\zeta}_r(t) m^*_{rr} + \zeta_r k^*_{rr} = -2\zeta_1^3(t) b^*_{111r} \quad \text{for } r=2,\ldots,n \tag{26}$$

appear as set of linear differential equations in ζr, with a right hand side forcing term, due to the non-linearity given by: $f_r(t) = -2b^*_{111r}\zeta_1^3(t)$.

The procedure adopted here was to solve first the first equation, i.e.

$$\ddot{\zeta}_{11}(t) m^*_{11} + \zeta_1 k^*_{11} + 2\zeta_1^3(t) b^*_{1111} = 0 \tag{27}$$

assuming a solution of the form:

$$\zeta_1(t) = a_1(\omega)\sin\omega t + a_1(3\omega)\sin 3\omega t \tag{28}$$

and using the harmonic balance method to calculate $a_1(\omega)$ and $a_1(3\omega)$. Then, this solution was injected in the linearised pseudo-forced system (26), which has been then directly solved by superposition, leading to the time functions $\zeta i(t)$, exhibiting the harmonic distortion due to the non-linearity.

3.2 Analytical results

Application of the procedure described above, has led for $a^1(3\omega)$ to:

$$a_1(3\omega) = \frac{1}{2}\frac{a_1^3(\omega) b^*_{1111}}{(9(k^*_{11} + a_1^2 b^*_{1111}) - k^*_{11})} \tag{29}$$

After substitution in (26), the solution obtained, restricted to the third harmonics for $\zeta i(t)$ was:

$$\zeta i(t) = \varepsilon_i(\omega)\sin(\omega t) + \varepsilon_i(3\omega)\sin(3\omega t) \tag{30}$$

where

$$\varepsilon_r(\omega) = -\frac{3a_1^3 b^*_{111r}}{2(-m^*_{rr}\omega^{*^2} + k^*_{rr})} \qquad \varepsilon_r(3\omega) = \frac{a_1^3 b^*_{r111}}{2(9(k^*_{11} + a_1^2 b^*_{1111}) - k^*_{rr})} \tag{31-32}$$

The first harmonic component of the non-linear free response, denoted as $W^*_\omega(x,t,a_1)$ can be explicitly expressed by the following series:

$$W^*_\omega(x,t,a_1)=\left[a_1 w^*_1(x)+\frac{3a_1^3 b^*_{1113}\, w^*_3(x)}{2((k^*_{11}+a_1^2 b^*_{1111})-k^*_{33})}+\cdots+\frac{3a_1^3 b^*_{1119}\, w^*_9(x)}{2((k^*_{11}+a_1^2 b^*_{1111})-k^*_{99})}+\frac{3a_1^3 b^*_{11111}\, w^*_{11}(x)}{2((k^*_{11}+a_1^2 b^*_{1111})-k^*_{1111})}\right]\sin\omega t \qquad (33)$$

The third harmonic component of the non-linear free response, denoted as $W^*_{3\omega}(x,t,a_1)$, can be explicitly expressed by the following series:

$$W^*_{3\omega}(x,t,a_1)=\left[-\frac{a_1^3 b^*_{1111} w^*_1(x)}{2(9(k^*_{11}+a_1^2 b^*_{1111})-k^*_{11})}+\frac{-a_1^3 b^*_{1113} w^*_3(x)}{2(9(k^*_{11}+a_1^2 b^*_{1111})-k^*_{33})}+\cdots+\frac{-a_1^3 b^*_{1119} w^*_9(x)}{2(9(k^*_{11}+a_1^2 b^*_{1111})-k^*_{99})}+\frac{-a_1^3 b^*_{11111} w^*_{11}(x)}{2(9(k^*_{11}+a_1^2 b^*_{1111})-k^*_{1111})}\right]\sin 3\omega t \qquad (34)$$

Expression (33) et (34) is an explicit simple formula, allowing direct calculation of the third harmonic contributions to the first non-linear beam mode shape, as functions of the assigned first mode contribution a_1 and of the known parameters k_{rr}, m_{rr} and b_{111r} computed numerically.

The results given in Table 1 correspond to the values of ε_1, ε_3, ε_5, .., ε_{11} corresponding to the third harmonic, obtained via the present new formulation for some assigned values of a_1 varying from 0.05 to 1.8, which corresponds to a maximum non-dimensional vibration amplitude at the beam centre varying from 0.0794 to 1.225. For each solution, the corresponding values of ω^*_{nl}/ω^*_l and the curvature calculated at x*=0 are also given. Comparison between Table 1, and data of table 2 taken from reference [9], based on another numerical approach, and the solution of a big set of numerical algebraic equations, shows that there exists a very good agreement between these results for finite amplitudes of vibration up to a displacement equal to the beam thickness, which corresponds to $a_1 \cong 0.67$. For higher values of the vibration amplitude, slight differences start to appear and increase with the amplitude of vibration, as can be seen in Figures 2-4.

Table 1. Explicit results obtained via the present formulation

$a_1(\omega)$	$\varepsilon_1(3\omega)$	$\varepsilon_3(3\omega)$	$\varepsilon_5(3\omega)$
0.05	-0.7061 E-5	-0.22 E-5	-0.20 E-6
0.3	-0.1401 E-2	-0.49 E-3	-0.44 E-4
0.55	-0.7205 E-2	-0.33 E-2	-0.28 E-3
0.8	-0.1755 E-1	-0.12 E-1	-0.87 E-3
1.05	-0.3087 E-1	-0.37 E-1	-0.20 E-2
1.55	-0.6115 E-1	-0.23 E-1	-0.69 E-2

Table 2. Numerical results, taken from reference [2], based on the iterative solution of a non-linear algebraic system

$a_1(\omega)$	$a_1(3\omega)$	a3(3ω)	$a_5(3\omega)$
0.05	-0.7071 E-5	-0.2219 E-5	-0.2077 E-6
0.3	-0.1419 E-2	-0.4743 E-3	-0.4565 E-4
0.55	-0.7450 E-2	-0.2834 E-2	-0.2890 E-3
0.8	-0.1853 E-1	-0.8253 E-2	-0.9050 E-3
1.05	-0.3312 E-1	-0.1729 E-1	-0.2042 E-2
1.55	-0.6678 E-1	-0.4616 E-1	-0.6193 E-2

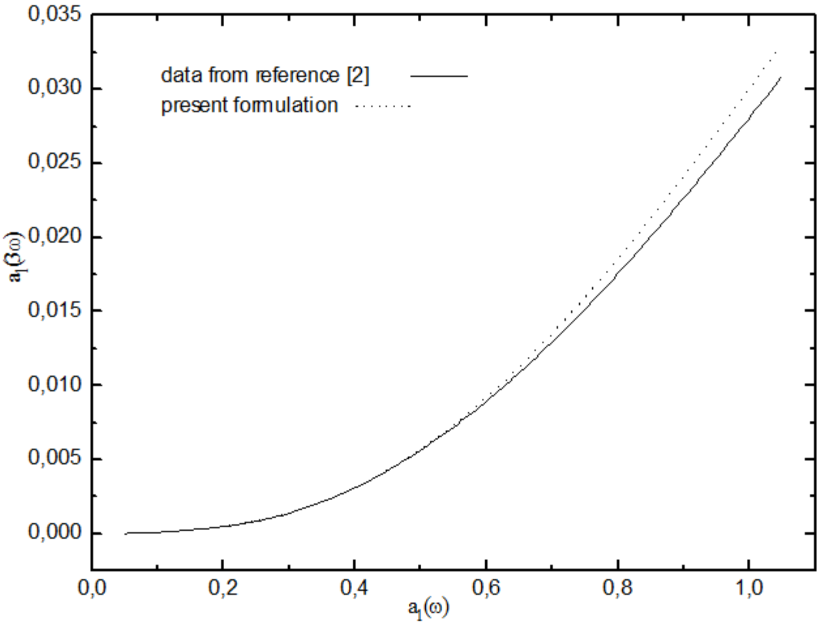

Fig. 2. Comparaison between the contribution $a_1(3\omega)$ of the data taken from reference [2] and the present formulation of the C-C beam

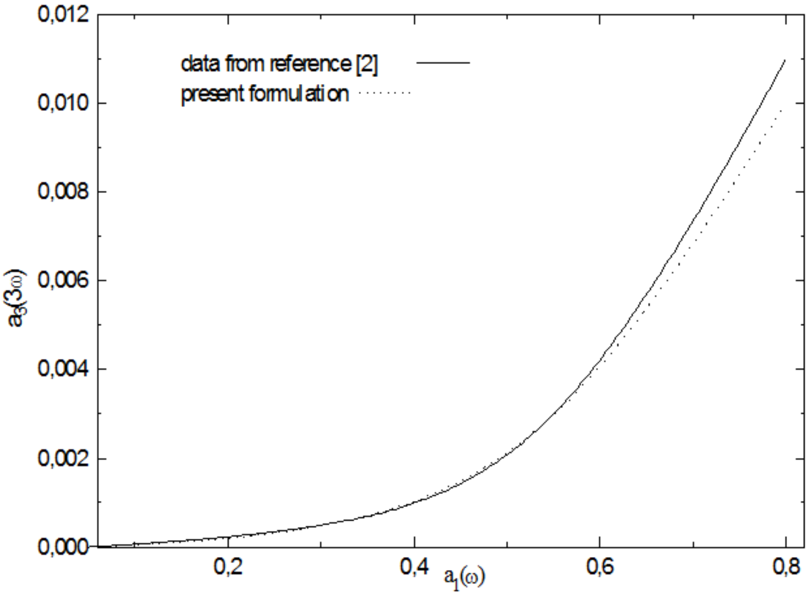

Fig. 3. Comparaison between the contribution $a_3(3\omega)$ of the data taken from reference [2] and the present formulation of the C-C beam

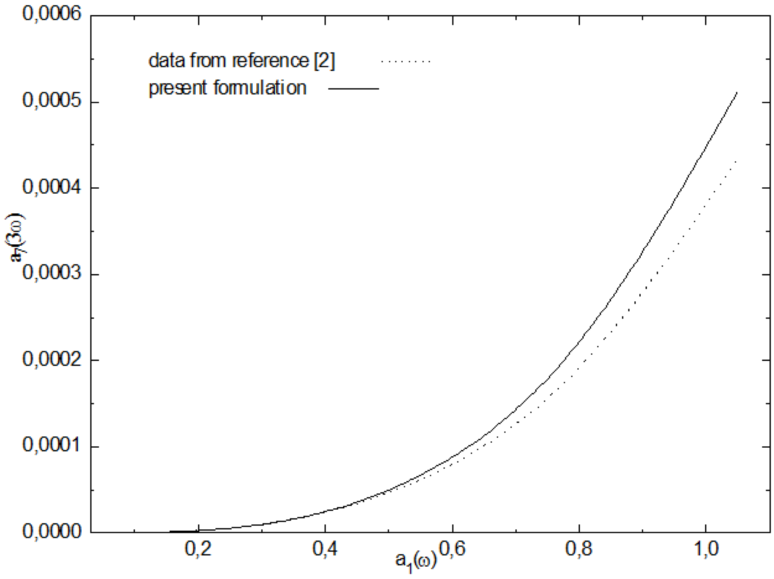

Fig. 4. Comparaison between the contribution $a_7(3\omega)$ of the data taken from reference [2] and the present formulation of the C-C beam

4 Conclusion

Considering the harmonic distortion occurring at large transverse vibration amplitudes, simple approximate analytical expressions for the higher mode contribution coefficients to the first and third harmonic of the response have been derived. It gives accurate results for moderate non-linearity, namely for vibration amplitudes up to 0.7 the beam thickness.

These analytical expressions for the harmonic distortion are expected to be very useful in developing new non-linear models for estimating the fatigue life of beams-like structures, undergoing high vibration amplitudes.

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