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On Composite Convolution Operators with Weight

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ABSTRACT: In this paper composite convolution operators with weight are introduced on Hilbert space H. Some basic properties for composite convolution operators with weight have been investigated. The characterization of normal, Hermitian and idempotent composite convolution operators with weight are explored. The commutant of composite convolution operators with weight has also been characterized.

KEYWORDS: Composite convolution operator with weight, Radon-Nikodym derivative, Expectation operator, idempotent, projection operator.

MATHEMATICS SUBJECT CLASSIFICATION: Primary 47B38; Secondary 47B99.

1 INTRODUCTION

Let (X,Ω,μ) be a σ-finite measure space and $\phi: X \to X$ be a non-singular measurable transformation ($\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0$). Then a composition transformation , for $1 \le p < \infty$, $C_\phi : L^p(\mu) \to L^p(\mu)$ is defined by $C_\phi f = f o \phi$, for every $f \in L^p(\mu)$. In case C_ϕ is continuous, we call it a composition operator induced by ϕ. It is easy to see that C_ϕ is a bounded operator if and only if $\frac{d\mu\phi^{-1}}{d\mu} = f_o$, the Radon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure μ , is essentially bounded. For more detail about composition operator and weighted composition operators, we refer to Singh and Manhas [10], Campbell [2] and Takagi [13]. For each $f \in L^p(\mu)$, $1 \le p < \infty$, there exists a unique $\phi^{-1}(\Omega)$ measurable function E(f) such that $\int g f \, d\mu = \int g E(f) \, d\mu$, for every $\phi^{-1}(\Omega)$ measurable function g for which left integral exists. The function E(f) is called Conditional Expectation of f with respect to the sub- algebra $\phi^{-1}(\Omega)$. For more properties of the expectation operator, we refer to Parthasarthy [9].

Given $f, g \in L^2(R)$, then convolution of f and g, $f*g$ can be defined by

$$f*g(x) = \int g(x-y)f(y)\,d\mu(y),$$

where g is fixed, $k(x,y) = g(x-y)$ is a convolution kernel, and the integral operator defined by

$$Wf(x) = \int k(x-y)f(y)\,d\mu(y)$$

is known as Convolution operator. Suppose $\phi : [0,1] \to [0,1]$ is a measurable transformation, then

$$W_\phi f(x) = \int k(x-y)f(\phi(y))d\mu(y)$$

$$= \int k_\phi(x-y)\, f(y)d\mu(y)$$

is known as composite convolution operator induced by pair (k,ϕ),

where $k_\phi(x\text{-}y) = E(f_o(y)k(x\text{-}y)\,\phi^{-1}(y))$.

Suppose $u : X \to \mathbb{C}$ is a measurable function. Then the bounded operator $W_{u,\phi}$ defined by

$$W_{u,\phi} f(x) = \int u(x)k(x\text{-}y)f(\phi(y))d\mu(y)$$

$$= \int k_{u,\phi}(x\text{-}y)\, f(y)d\mu(y),$$

is known as composite convolution operators with weight,

where $k_{u,\phi}(x\text{-}y) = u(x)E(k(x\text{-}\,\phi^{-1}(y))\, f_o(y)) = u(x)E(k_x(\phi^{-1}(y))\, f_o(y))$

It is easy to observe that $W_{u,\phi} = W_\phi$ if $u(x) = 1$ for every $x \in X$, and $W = W_\phi$, if $\phi(x) = x$ for all $x \in X$. Also, it is clear that $W_{u,\phi} = M_u W_\phi$, where M_u is a multiplication operator induced by u.

The symbol $L^p(\mu)$ denotes the collection of all measurable functions $f : X \to R$ (or C) such that

$(\int_X |f(x)|^p d\mu)^{1/p} < \infty$. The space $L^p(X, S, \mu)$ is a Banach space under the norm defined by $\|f\|_p = (\int_X |f|^p d\mu)^{1/p}$.

If p = 2, then $L^2(\mu)$, the space of square-integrable functions of complex numbers is a Hilbert space. By $B(L^2(\mu))$, we denote the Banach space of all bounded linear operators from $L^2(\mu)$ into itself.

For literature related to integral operators in particular on convolution operators we refer to Stepanov ([11],[12]), Bloom and Kerman [1], Halmos and Sunder [7], Lybic's [8]. Gupta and Komal ([3],[4]), Gupta ([5],[6]) also studied composite integral operators and composite convolution operators. In this paper the study of composite convolution operators with weight is initiated. The adjoint of composite convolution operator with weight has been obtained. Hermitian, normal, idempotent and bounded composite convolution operators with weight are characterized. An attempt has been made to explore the commutant of composite convolution operator with weight. It is also shown that the set of all composite convolution operators with weight is algebra

2 HERMITIAN, NORMAL AND IDEMPOTENT COMPOSITE CONVOLUTION OPERATORS WITH WEIGHT

In this section a sufficient condition for bounded composite convolution operator with weight on $L^2(\mu)$ has been obtained. The criterion for composite convolution operator with weight to be Hermitian, normal and idempotent have been discussed.

Theorem 2.1: Let $k_{u,\phi} \in L^2(\mu\times\mu)$. Then $W_{u,\phi}$ is a bounded composite convolution operator with weight.

Proof: For every $f \in L^2(\mu)$, we have

$$\|W_{u,\phi}f\|^2 = \int_X \left|\int_X u(x)\, k(x\text{-}y)f(\phi(y))d\mu(y)\right|^2 d\mu(x)$$

$$= \int_X \left|\int_X u(x)E(k_x(\phi^{-1}(y))f_o(y))f(y)d\mu(y)\right|^2 d\mu(x)$$

$$\le \int_X \int_X |u(x)\, f_o(y)\, E(k_x o\phi^{-1}(y))|^2 d\mu(x)\, d\mu(y)(\int |f(y)|^2 d\mu(y))$$

$$= \left(\int_X \int_X |k_{u,\phi}(x - y)|^2 d\mu(x)\, d\mu(y)\right).\, \|f\|^2$$

$$= \|k_{u,\phi}\|^2 .\, \|f\|^2$$

Hence $W_{u,\phi}$ is a bounded operator.

In the following theorem we compute the adjoint of a composite convolution operator with weight.

Let $A: L^2(\mu) \to L^2(\mu)$ be defined by

$$(Af)(x) = \int f_o(y\, E((uf)o\phi^{-1})(y)\, \overline{k(y-x)}\, d\mu(y)$$

Theorem 2.2: If $W_{u,\phi} \in B(L^2(\mu))$, then $A = W^*_{u,\phi}$.

Proof: Consider

$$\langle f, W_{u,\phi} g \rangle = \int f(x) \int \overline{u(x)k(x-y)g(\phi(y))}\, d\mu(y)\, d\mu(x).$$

$$= \iint f_o(x)(E(ufo\phi^{-1})(x)\, \overline{k(x-y)}\, d\mu(x)\, \overline{g}(y)\, d\mu(y),$$

(By using Fubini's Theorem)

$$= \int (A f)(y)\, \overline{g}(y)\, d\mu(y).$$

$$= \langle A f, g \rangle \quad \text{for every } f, g \in L^2(\mu).$$

Hence $A = W^*_{u,\phi}$. This completes the proof of theorem.

Theorem 2.3: Let $W_{u,\phi} \in B(L^2(\mu))$. Suppose $\phi^{-1}(S) = S$. Then $W_{u,\phi}$ is Hermitian if and only if

$$\int_E \int_{\phi^{-1}(F)} f_o(x)\, k(x - y)\, E(u \circ \phi^{-1})(x) d\mu(x)\, d\mu(y)$$

$$= \int_{\phi^{-1}(E)} \int_F f_o(y)\, \overline{k(y-x)}\, E(uo\phi^{-1})(y) d\mu(y) d\mu(x).$$

Proof: Suppose $W_{u,\phi}$ is Hermitian. Then for any measurable rectangle $E \times F$ of finite measure,

$$\langle W^*_{u,\phi}\, \chi_{\phi^{-1}(E)}, \chi_{\phi^{-1}(F)} \rangle = \iint f_o(y)\, \overline{k(y-x)}\, E(u\, \chi_{\phi^{-1}(E)} \circ \phi^{-1}(y))\, d\mu(y)\, \chi_{\phi^{-1}(F)}(x) d\mu(x)$$

$$= \iint f_o(y)\, \overline{k(y-x)}\, E(u\, \chi_E \circ \phi^{-1})\, (y) d\mu(y)\, \chi_{\phi^{-1}(F)}(x)\, d\mu(x)$$

$$= \int_E \int_{\phi^{-1}(F)} f_o(y)\, \overline{k(y-x)}\, E(uo\, \phi^{-1})(y) d\mu(y)\, d\mu(x) \tag{2.1}$$

and

$$\langle W_{u,\phi}\, \chi_{\phi^{-1}(E)}, \chi_{\phi^{-1}(F)} \rangle = \iint u(x)k(x-y)\, \chi_{\phi^{-1}(E)}(y)\, d\mu(y)\, \chi_{\phi^{-1}(F)}(x)\, d\mu(x)$$

$$= \iint f_o(y)k(x-y)E(uo\phi^{-1})(x)\chi_{(F)}(y)\, d\mu(y)\, \chi_{\phi^{-1}(E)}(x)\, d\mu(x)$$

$$= \iint f_o(y)k(x-y)E(uo\phi^{-1})(x)\chi_{(F)}(y)\, d\mu(y)\, \chi_{\phi^{-1}(E)}(x) d\mu(x)$$

$$= \int_{\phi^{-1}(E)} \int_F f_o(y)k(x-y)E(uo\phi^{-1})(x) d\mu(x) d\mu(y) \tag{2.2}$$

From the equations (2.1) and (2.2), it is clear that if $W_{u,\phi}$ is Hermitian, then the condition must hold.

Conversely, if the condition is true, then clearly

$$\langle W^*_{u,\phi}\, \chi_{\phi^{-1}(E)}, g \rangle = \langle W_{u,\phi}\, \chi_{\phi^{-1}(E)}, g \rangle \tag{2.3}$$

for every simple function g. Since simple functions are dense in $L^p(\mu)$, so the equation (2.3) is true for every $g \in L^2(\mu)$.

This proves that $W_{u,\phi}$ is a Hermitian operator.

Theorem 2.4: Let $W_{u,\phi} \in B(L^2(\mu))$. Then $W_{u,\phi}$ is normal if and only if

$$\int_X k^*_{u,\phi}(x-y)\, k_{u,\phi}(y-z)d\mu(z)d\mu(y) = \int_X k_{u,\phi}(x-y)\, k^*_{u,\phi}(y-z)d\mu(z)d\mu(y).$$

Proof: Suppose $W_{u,\phi}$ is normal. For any measurable rectangle $E\times F$ of finite measure, we have

$$\langle W^*_{u,\phi} W_{u,\phi}\chi_E, \chi_F\rangle = \int_X \iint k^*_{u,\phi}(x-y)\, k_{u,\phi}(y-z)\chi_E(z)\,\chi_F(x)d\mu(z)d\mu(y)d\mu(x)$$

$$= \int_X \iint_{E\times F} k^*_{u,\phi}(x-y)\, k_{u,\phi}(y-z)d\mu(z)d\mu(y)d\mu(x)$$

and $\langle W_{u,\phi} W^*_{u,\phi}\chi_E, \chi_F\rangle$

$$= \int_X \iint k_{u,\phi}(x-y)\, k^*_{u,\phi}(y-z)\chi_E(z)\,\chi_F(x)d\mu(z)d\mu(y)d\mu(x)$$

$$= \int_X \iint_{E\times F} k_{u,\phi}(x-y)\, k^*_{u,\phi}(y-z)d\mu(z)d\mu(y)d\mu(x).$$

Thus, it follows that

$$\int_X k^*_{u,\phi}(x-y)\, k_{u,\phi}(y-z)d\mu(z)d\mu(y) = \int_X k_{u,\phi}(x-y)\, k^*_{u,\phi}(y-z)d\mu(z)d\mu(y).$$

Conversely, if the condition is true then it is obvious that $W_{u,\phi}$ is a normal operator.

Thus the proof of the theorem is complete.

Theorem 2.5: Let $W_{u,\phi}\in B(L^2(\mu))$. Then $W_{u,\phi}$ is idempotent if and only if kernel $k_{u,\phi}$ is idempotent.

Proof: If the condition is true , then an easy computation shows that $W_{u,\phi}$ is idempotent.

For $f, g \in L^2(\mu)$, we have

$$\langle W^2_{u,\phi} f, g\rangle = \langle W_{u,\phi} f, W^*_{u,\phi}, g\rangle$$

$$= \int_X \left(\int u(x)k(x-y)f(\phi(y))d\mu(y)\right).\int \overline{u(x)k^*(x-z)g(\phi(z))d\mu(z)}\, d\mu(x)$$

$$= \int_X \iint k_{u,\phi}(x-y)f(y)\, k_{u,\phi}(z-x)\,\overline{g}(z)d\mu(z)\, d\mu(y)\, d\mu(x)$$

$$= \int_X \iint k_{u,\phi}(z-x)\, k_{u,\phi}(x-y)\, d\mu(x)f(y)\, d\mu(y)\,\overline{g}(z)d\mu(z)$$

$$= \int W_{u,\phi} f(z)\,\overline{g}(z)\, d\mu(z) = \langle W_{u,\phi} f, g\rangle$$

(where $k^2_{u,\phi}(z-y) = \int k_{u,\phi}(z-x)\, k_{u,\phi}(x-y)\, d\mu(x)$, as $k_{u,\phi}$ is idempotent, so $k_{u,\phi} = k^2_{u,\phi}$)

Hence $W_{u,\phi}$ is an idempotent operator.

Conversely, if $W_{u,\phi}$ is an idempotent operator, then taking $f = \chi_E$ and $g = \chi_F$.

$$\langle W_{u,\phi}^2 \chi_E, \chi_F \rangle = \langle W_{u,\phi} \chi_E, \chi_F \rangle$$

i.e., $$\int \int_E \int_F k_{u,\phi}(z-x)\, k_{u,\phi}(x-y)\, d\mu(x)\, d\mu(y) d\mu(z) = \int \int_{E\times F} k_{u,\phi}(z-y)\, d\mu(y)\, d\mu(z)$$

Thus

$$\int_{E\times F} k_{u,\phi}^2(z-y)\, d\mu(y) = \int_{E\times F} k_{u,\phi}(z-y) d\mu(y),$$ which proves that $k_{u,\phi}$ is idempotent.

Corollary 2.6: Let $W_{u,\phi} \in B(L^2(\mu))$. Then $W_{u,\phi}$ is an projection if and only if $k_{u,\phi}$ is an idempotent and

$\overline{k_{u,\phi}(y-x)} = k_{u,\phi}(x-y)$.

In the next result commutant of composite convolution operators with weight is obtained.

Theorem 2.7: Let $W_{u,\phi} \in B(L^2(\mu))$. Then M_θ commutes with $W_{u,\phi}$ if and only if $\theta = \theta o \phi$ a.e.

Proof: For $f \in L^p(\mu)$,

$$(W_{u,\phi} M_\theta f)(x) = \int u(x)\, k(x-y)(M_\theta f) o\phi(y)\, d\mu(y)$$

$$= \int u(x)\, k(x-y)\, \theta(\phi(y))\, f(\phi(y))\, d\mu(y) \tag{2.4}$$

and $(M_\theta W_{u,\phi} f)(x) = \theta(x)(W_{u,\phi} f)(x)$

$$= \theta(x) \int u(x)\, k(x-y)\, f(\phi(y) d\mu(y) \tag{2.5}$$

In view of equations (2.4) and (2.5), we have

$$(M_\theta W_{u,\phi} f)(x) - (W_{u,\phi} M_\theta f)(x) = \int u(x)\, k(x-y)\, [\theta(\phi(y)) - \theta(x)]\, f(\phi(y))\, d\mu(y).$$

Hence, the result.

Theorem 2.8: Let $S = \{ W_{u,\phi} : W_{u,\phi} \in B(L^2(\mu))\}$. Then S is an algebra of $B(L^2(\mu)$.

Proof: Let $W_{u,\phi}$ and $I_{u,\phi}$ be composite convolution operators with weight induced by kernels k and h respectively and $W_{u,\phi}$, $I_{u,\phi} \in B(L^2(\mu))$.

Then, for $f \in L^2(\mu)$, we have

$$(W_{u,\phi} + I_{u,\phi})\, f(x) = W_{u,\phi} f(x) + I_{u,\phi} f(x)$$

$$= \int_X u(x) k(x-y)\, f(\phi(y))\, d\mu(y) + \int_X u(x) h(x-y)\, f(\phi(y))\, d\mu(y)$$

$$= \int_X u(x)\, [\, k(x-y) + h(x-y)\,]\, f(\phi(y))\, d\mu(y)$$

$$= A_{u,\phi} f(x),$$

where $A_{u,\phi}$ is composite convolution operators with weight induced by kernels (k+h) .

Also, for any scalar α, we have

$$(W_{u,\phi}\, \alpha f)(x) = \int_X u(x) k(x-y)\, \alpha\, f(\phi(y))\, d\mu(y)$$

$$= \int_X \alpha\, u(x)k(x-y)f(\phi(y))\, d\mu(y)$$

$$= \alpha\, W_{u,\phi} f(x).$$

Hence, the result proved.

Theorem 2.9: The product of two composite convolution operators ($W_{u,\phi} . I_{u,\phi}$) is a composite convolution operators $I_{k_{u,\phi}h_{u,\phi}}$, if $(k_{u,\phi} h_{u,\phi})(x-z) = \int k_{u,\phi}(x-y)\, h_{u,\phi}(y-z)d\mu(y)$.

Moreover, $W_{u,\phi}\ I_{u,\phi} = I_{u,\phi}\, W_{u,\phi}$, if $(k_{u,\phi} h_{u,\phi})(x-z) = (h_{u,\phi} k_{u,\phi})(x-z)$

Proof: For every $f \in L^2[0,1]$,

$$W_{u,\phi}\ I_{u,\phi} f(x) = \int k_{u,\phi}(x-y)\, (I_{u,\phi}f)(y)d\mu(y)$$

$$= \int\int k_{u,\phi}(x-y)\, h_{u,\phi}(y-z)\, f(z)\, d\mu(z)\, d\mu(y)$$

$$= \int \left[\int k_{u,\phi}(x-y)\, h_{u,\phi}(y-z)d\mu(y)\right] f(z)\, d\mu(z)$$

$$= \int (k_{u,\phi} h_{u,\phi})(x-z)\, f(z)d\mu(z) \qquad (2.6)$$

Hence, the product of two composite convolution operators with weight, $W_{u,\phi} . I_{u,\phi}$ is again a composite convolution operator with weight $I_{k_{u,\phi}h_{u,\phi}}$ induced by convolution kernel $k_{u,\phi}h_{u,\phi}$.

Again, $I_{u,\phi}\, W_{u,\phi} f(x) = \int\int h_{u,\phi}(x-y)k_{u,\phi}(y-z)\, f(z)\, d\mu(z)\, d\mu(y)$

$$= \int (h_{u,\phi} k_{u,\phi})(x-z)\, f(z)d\mu(z) \qquad (2.7)$$

The equations (2.6) and (2.7) give desired conclusion.

Corollary 3.5: The product of two composite convolution operators is zero if at least one of them is zero.

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