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An analytical solution of a problem of a tubular structure which is subjected to an expansion following the axis

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ABSTRACT: In this study, we propose the analysis of a tubular structure undergoing expansion along the axis of the cylindrical material under internal pressure. Both kinematic and dynamic aspects are examined, leading to the derivation of an exact analytical solution using a system of partial differential equations. Simulation results demonstrate that the solution exhibits sinusoidal behavior in all cases. Minor variations result in incremental or decremental intervals, while significant changes in radius cause simultaneous increase and decrease intervals with trigonometric patterns. Additionally, we observe that the second component significantly influences the overall solution behavior compared to the first component.

KEYWORDS: Kinematics of transformation, gradient tensor, Cauchy-Green tensor, isotropic elementaries invariants, incompressible transformation, equations of equilibrium, Bessel differential equations solutions.

1 INTRODUCTION

In recent years, researchers seem better equipped to discuss the different methods of finding analytical solutions in mathematics, physics or elasticity. This is the case, for example, for irregular problems such as large-scale bodies or areas, where analytical solutions are often found. However, these must be considered as approximate since they are valid only «far from certain edges» that is to say, the edges where are laid conditions to the overall limits of the solid body considered according to the Saint Venant' principle. Analytical solutions provide a better understanding of the essential characteristics of finite transformations. The usual reflexes of superposition of solutions and intuition resulting from linearity or nonlinearity must be abandoned in favor of a complete and rigorous approach of analysis of transformation and behavior. In addition, the choice of a model of behavior of a real material is not always easy. The behaviour of real materials is often complex. Even for structures as common as steel, many aspects of behaviour remain poorly understood and it is even difficult to develop a model representing the behaviour of a given material in all circumstances. In each mechanical or physical problem, it is necessary to choose the simplest model leading to satisfactory results for the intended use. Nowadays, research may have been inhibited by the belief that no progress could be made in the development of theories of nonlinear models unless a completely explicit constitutive equation could be written. Such equations were generally chosen on the basis of an alleged simplicity of constitutive equations. One of the difficulties of this approach lies in the fact that simplicity is very subjective, depending considerably on the choice of variables according to which the relationship is expressed. The transition to the constitutive equation, expressed in phenomenological terms, is generally very difficult. It cannot be done without a related model and complex mathematical considerations. The most modern approach stems largely from the realization that it is possible to write fairly general constitutive equations from phenomenological or geometric considerations. This awareness is already involved in the theory of finite elasticity. Here, the constitutive equation is given by a statement that the strain energy function must depend on the strain gradient. In this paper, we propose the study of the behavior of a structure in tubular form. It is subjected to an expansion following the axis of the material that we assume cylindrical and to an internal pressure. The first part of this study concerns the kinematic and dynamic aspects related to the behavior of the model. We will then proceed to an analytical solution, exact of the problem at the limits resulting from the setting of equations through a system of partial differential equations. We will then simulate the different components of the solution and the solution due to pressure.

2 MATHEMATICAL CONSIDERATIONS

A tube of circular section, of axis $\vec{E}_z$ of inner radius R_i of outer radius R_o and of thickness$H = \frac{R_o - R_i}{2}$.

It is subjected to an $\vec{E}_z$ axis expansion and an P internal pressure.

Under the action of these stresses, it is considered that the initial tube is transformed into a circular base cylinder, of axis $\vec{e}_z$ inner radiusr_i, outer radius r_o and thickness$h$.

The kinematics of the transformation, kinematically permissible, is defined as:

$$r = \alpha R, \theta = \beta \Theta, z = \lambda Z + f(R,t) \tag{1}$$

The F gradient of this transformation and the left Cauchy-Green tensor$B = FF^T$ take the forms:

$$F = \begin{pmatrix} \alpha & \frac{-\theta}{R} & 0 \\ 0 & 1+\frac{\beta}{R} & 0 \\ \frac{\partial f(R,t)}{\partial R} & 0 & \lambda \end{pmatrix}, B = \begin{pmatrix} \alpha^2 + \frac{\theta^2}{R^2} & \frac{-\theta}{R}\left(1+\frac{\beta}{R}\right) & \alpha \frac{\partial f(R,t)}{\partial R} \\ \frac{-\theta}{R}\left(1+\frac{\beta}{R}\right) & \left(1+\frac{\beta}{R}\right)^2 & 0 \\ \alpha \frac{\partial f(R,t)}{\partial R} & 0 & \lambda^2 + \left(\frac{\partial f(R,t)}{\partial R}\right)^2 \end{pmatrix} \tag{2}$$

To obtain a behavior relationship that describes the nonlinear hyperelastic mechanical behavior is to define a behavior relationship linking constraints and deformations. To do this, Spencer introduced the second symmetrical Lagrangian tensor of Piola-Kirchoff constraints [1]. From the kinematic data associated with the transformation, the Eulerian tensor deformation of Cauchy-Green dilations was characterized. The energy potential depends on the deformation invariants: $W = W(I_1, I_2, I_3)$.

The $I_j, j = 1,2,3$are the three elementary invariants of tensor B defined by:

$$\begin{cases} I_1 = trace(B) \\ I_3 = \det(B) \\ I_2 = I_3 . trace(B^{-1}) \end{cases} \tag{3}$$

The stress state for an incompressible isotropic hyperelastic behaviour of energy W is written:

$$\sigma = \frac{2}{J}[W_1 B + W_2(I_1 I_d - B) + W_3 I_1 I_d] \tag{4}$$

where I_d is the identity matrix of order 3,$W_j = \frac{\partial W}{\partial I_j}, j = 1,2,3$.

Considering the equalities (1), (2) and (3), the components of the Cauchy stress tensor, in a system of cylindrical coordinates and considering the nature of the kinematics defined in (1) and the components of the Cauchy tensor (4), the equations of motion are reduced to the system:

$$\frac{2(W_2+W_3)}{\alpha}\frac{\partial f(R,t)}{\partial R}\frac{\partial^2 f(R,t)}{\partial R^2} + \left[\frac{-\theta^2(W_2+W_3)}{\alpha} - 2\frac{\beta^2(W_2+W_3)}{\alpha} - \frac{(W_1-W_2)}{\lambda} + \frac{(\theta^2-\beta^2)(W_1+W_2)}{\lambda}\right]\frac{1}{R^3} + \left[-2\frac{\beta(W_2+W_3)}{\alpha} - \frac{(W_1-W_2)}{\lambda\beta} - 2\frac{\beta(W_1+W_2)}{\lambda}\right]\frac{1}{R^2} + \left[\frac{(W_1+W_2)(\alpha^2-1)}{\lambda R}\right] = 0 \tag{5}$$

$$\frac{\partial^2 f(R,t)}{\partial R^2} + \frac{1}{R}\frac{\partial f(R,t)}{\partial R} - \frac{\alpha\lambda\rho_o}{W_1-W_2}\left(1+\frac{\beta}{R}\right)\frac{\partial^2 f(R,t)}{\partial t^2} = 0 \tag{6}$$

The main problem that concerns us when dealing with a differential equation is the search (analytical, numerical or approximate) for its solutions. In theory of differential equations as in algebra one can interpret in different ways. In algebra we tried to find a general formula, using radicals [2]. There is the possibility of looking for the approximate solution of equations with numerical coefficients, as well as that of studying the dependence of solutions on their coefficients. In this paragraph [3], we use this last possibility with the method of decoupled equations by posing:

$$f(R,t) = U(R) + V(R)\cos(wt) \tag{7}$$

By reporting equation (7) in (6), we obtain Bessel equations (first and second types)

There are different methods to solve the Bessel differential equation [4], in this article we use the Laplace transformation.

$$\begin{cases} U''(R) + \frac{1}{R}U'(R) = 0 \\ V''(R) + \frac{1}{R}V'(R) + \frac{\alpha\lambda w^2 \rho_0}{W_1 - W_2}\left(1 + \frac{\beta}{R}\right)V(R) = 0 \end{cases} \quad (8)$$

We get analytical solutions in the form of power series. The solutions of these equations are Bessel functions, the importance of which we know in mechanics [5,6,7,8].

Equation (8.2) is a modified Bessel equation whose solution is a series of functions. We then obtain the analytical solutions of (8):

$$\begin{cases} U(R) = A_0 \ln(R) + A_1 \\ V(R) = A_2 J_\nu(kR) + A_3 Y_\nu(kR) \\ J_\nu(kR) = \left(\frac{kR}{2}\right)^\nu \sum_r \frac{(-1)^r}{r!\Gamma(1+r+\nu)}\left(\frac{kR}{2}\right)^{2r} \\ Y_\nu(kR) = \frac{\cos(\nu\pi)J_\nu(kR) - J_{-\nu}(kR)}{\sin(\nu\pi)} \end{cases} \quad (9)$$

with

$k^2 = \frac{\alpha\lambda w^2 \rho_0}{W_1 - W_2}$, $\nu^2 = \frac{\alpha\beta\lambda w^2 \rho_0}{W_2 - W_1}$,$A_i$, $i = 0,1,2,3$ are intégration constants.

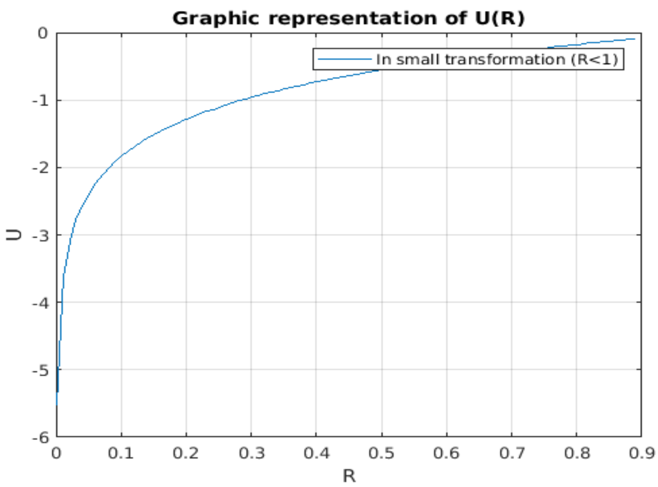
Thin wall pressure cylinders have wide use in different industrial applications like liquid storage tanks. The knowledge of the deformations in the thin walls and the stresses generated is an essential condition for the dimensioning of these structures. Before deformation, the initial internal pressure P_0 acts on the surface of the tube wall and the stresses are distributed in a non-uniform way over the section. We impose as conditions on the external lateral surface, radial and longitudinal stress [9].

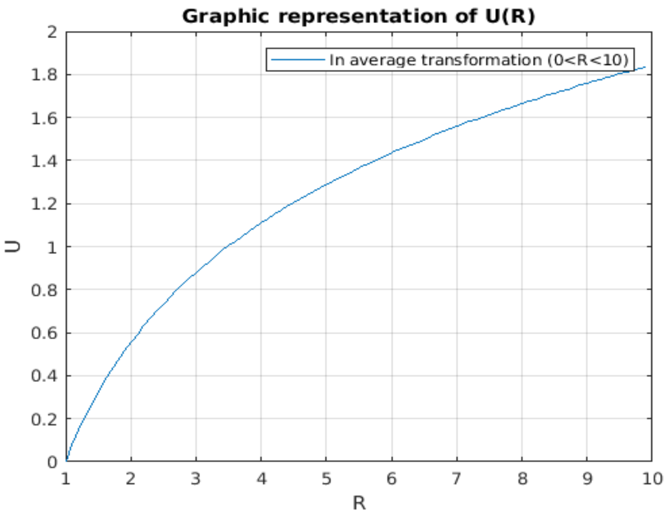
$$\begin{cases} f(R_o, 0) = 0 \\ \sigma_{rr}(R_i, 0) = -P_0\left(\left(\frac{R_0}{R_i}\right)^2 - 1\right) \\ \sigma_{zz}(R_o, 0) = P_0\left(\left(\frac{R_0}{R_i}\right)^2 - 1\right) \end{cases} \quad (10)$$

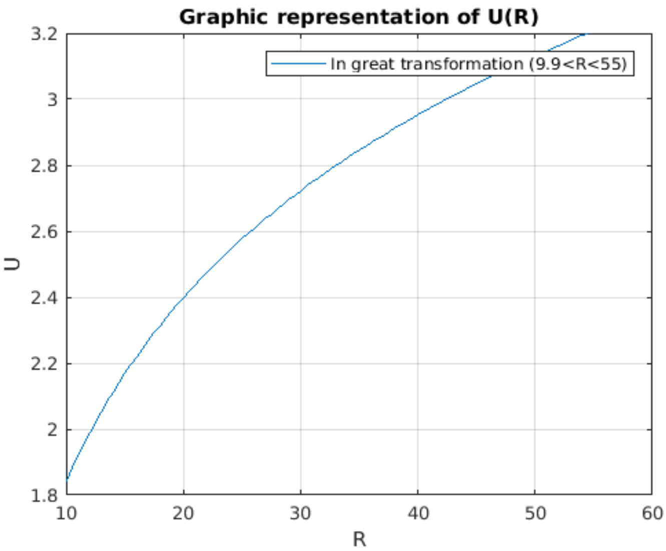
3 Simulation And Interpretation

Here we consider a cylindrical section with small, average and great radius to see the behavior of the solution $f(R,t)$.

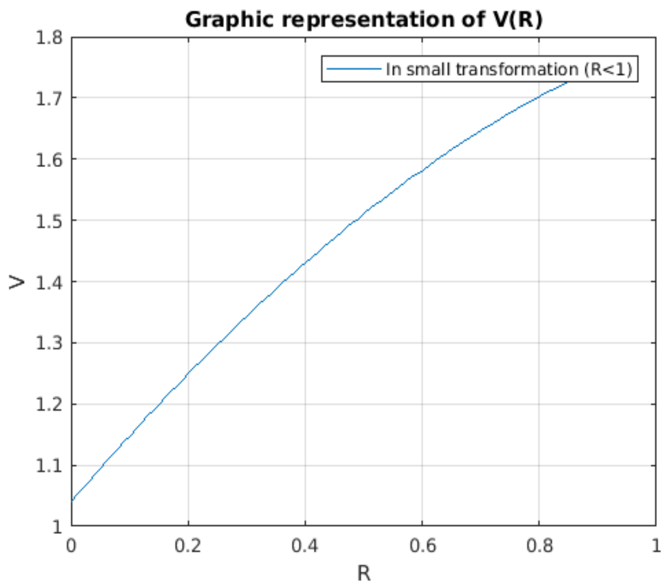
3.1 Function $U(R)$



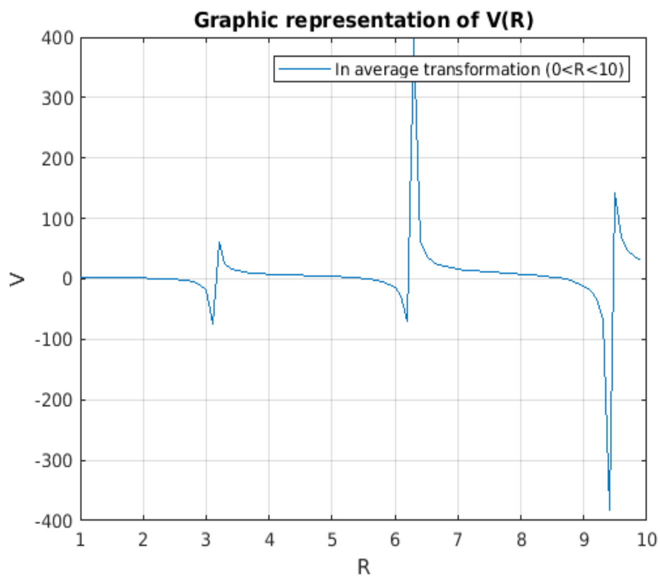


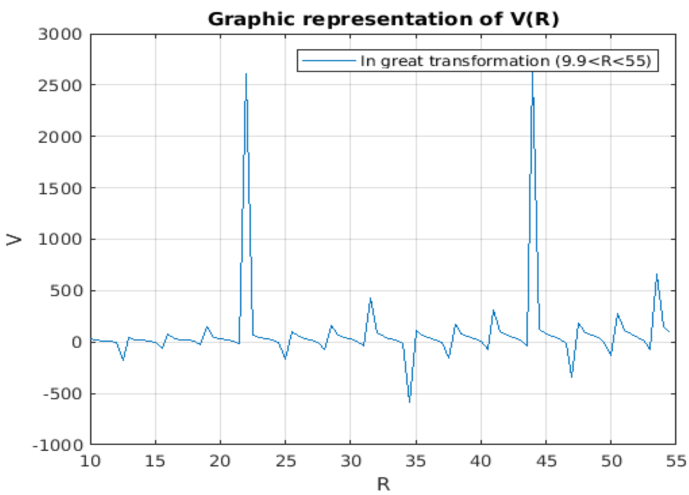


Here we can see that the kind of transformation have no influence in the behavior of the solution $U(R)$, in all the cases this component stays in logarithmic increasing.



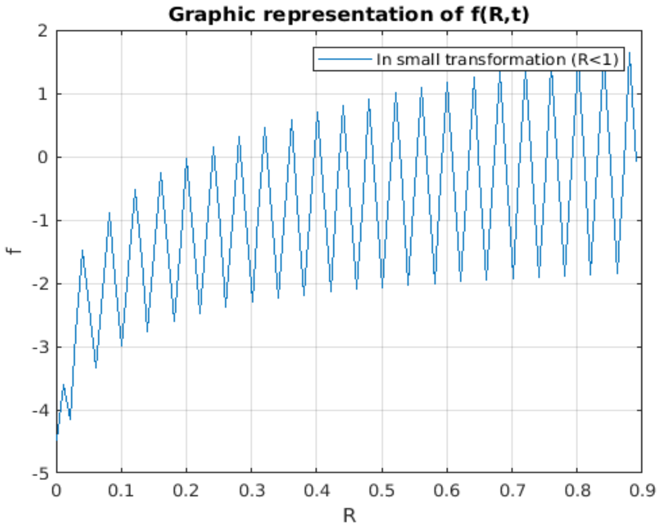
3.2 FUNCTION $V(R)$

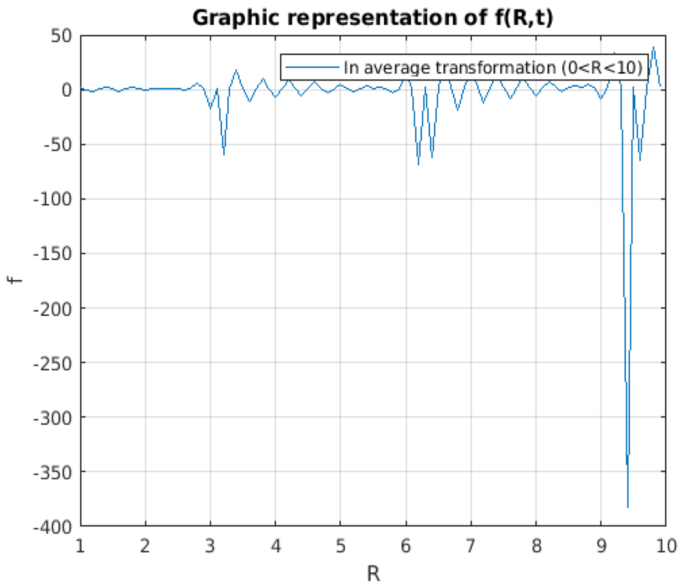


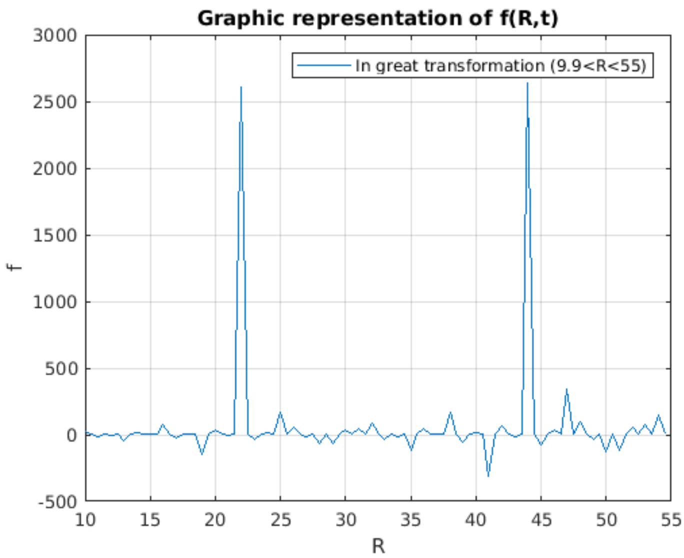


In this component $V(R)$, we can see that the kind of transformation has a big influence in the behavior of this one. That can be explain by when the variation of the radius is so small we stay in an interval of increasing or decreasing depending to the values of R, but when the variation of the radius is so great we stay in an interval of increasing and decreasing at same time with a trigonometric behavior depending to the values of R.

3.3 FUNCTION $f(R,t)$







The simulation of the problem solution $f(R,t)$ shows in all the cases that it remains in sunisoidal behavior. As in the following case, we can see that the kind of transformation has a big influence in the behavior of this one. That can be explain by when the variation of the radius is so small we stay in an interval of increasing or decreasing depending to the values of R, but when the variation of the radius is so great we stay in an interval of increasing and decreasing at same time with a trigonometric behavior depending to the values of R.

As a conclusion, we can say that the second component has a big influence in the behavior of the solution in general compared to the first component.

4 Conclusion

In this paper, we proposed the study of the behavior of a structure in tubular form which is subjected to an expansion following the axis of the material that we assumed cylindrical and to an internal pressure. In the first part, kinematic and dynamic aspects related to the behavior of the model have been studied. We proceeded to an exact analytical solution of the problem at the limits resulting from the setting of equations through a system of partial differential equations has been found. Simulation of the different components of the solution and the solution due to pressure shows that: $U(R)$ stays in all the cases this component in logarithmic increasing. $V(R)$, shows a big influence in the behavior of this one. when the variation of the radius is so small we stay in an interval of increasing or decreasing depending to the values of R, but when the variation of the radius is so great we stay in an interval of increasing and decreasing at same time with a trigonometric behavior depending to the values of R. $f(R,t)$ shows in all the cases that it remains in sunisoidal behavior. As in the following case, transformation has a big influence in the behavior of this one, when the variation of the radius is so small we stay in an interval of increasing or decreasing, but when the variation of the radius is so great we stay in an interval of increasing and decreasing at same time with a trigonometric behavior. And the second component has a big influence in the behavior of the solution in general compared to the first component.

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