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Abelian Theorem for Generalized Fourier-Laplace Transform

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ABSTRACT: Integral transformations have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics, engineering and technology. The origin of the integral transforms is the Fourier and Laplace transforms. Integral transformation is one of the well known techniques used for function transformation. Integral transform methods have proved to be of great importance in initial and boundary value problems of partial differential equation. The Fourier as well as Laplace transforms have various applications in various fields like science, physics, mathematics, engineering, geophysics, medical, cellular phones, electronics and many more as we have seen in earlier papers. In this paper we discussed about Fourier-Laplace transforms and this Fourier-Laplace transform may also have many applications in various fields.
This paper presents the generalization of Fourier-Laplace transform in distributional sense. Testing function spaces are defined, also some Abelian theorems of the initial and final value type are given. And the Abelian theorems are of considerable importance in solving boundary value problems of mathematical physics.

KEYWORDS: Fourier transform, Laplace transform, Fourier- Laplace transform, generalized function, Testing function space.

1 INTRODUCTION

Transform methods are widely used in many areas of science and engineering. For example, transform methods are used in signal processing and circuit analysis, in applications of probability theory. The theory of integral transform has presented a direct and systematic technique for the resolution of certain type of classical boundary and initial value problems [1]. The origin of the integral transforms including the Laplace and Fourier transforms can be traced back to celebrated work of P. S. Laplace in 1749-1827 on probability theory in the 1780s and to monumental treatise of Joseph Fourier (1768-1830) on La Theorie Analytique de la Chaleur published in 1822 [2].

Many linear boundary value and initial value problems in applied mathematics, mathematical physics and engineering science can be effectively solved by the use of Fourier transform. Fourier and Laplace transforms have used in Mechanical networks consisting of springs, masses and dampers, for the production of shock absorbers for example, processes to analyze chemical components, optical systems, and computer programs to process digitized sounds or images, can all be considered as systems[3]. The Fourier as well as Laplace transforms have various applications in various fields like science, physics, mathematics, engineering, geophysics, medical, cellular phones, electronics and many more as we have seen in earlier papers.

In this paper we prove the Abelian theorems. The setting for these Abelian theorems is motivated by the initial and final value results of Doetsch [4]. In mathematical analysis, the initial value theorem is a theorem used to relate frequency domain expression to the time domain behavior as time approaches zero. And the final value theorem (FVT) is one of several similar theorems used to relate frequency domain expression to the time domain behavior as time approaches infinity. A final value theorem allows the time domain behavior to be directly calculated by taking a limit of a frequency domain expression, as opposed to converting to a time domain expression and taking its limit. The Abelian theorems are obtained as the complex variable of the transform approaches 0 or ∞ in absolute value inside a wedge region in the right half plane.

In the present paper we established the Abelian theorem for generalized Fourier-Laplace transform. In section 2, we have defined some testing function spaces. We have given the definitions in section 3. Section 4 is devoted the definition of distributional generalized Fourier-Laplace transform. Initial value theorem for generalized Fourier-Laplace transform is given in section 5. Finally in section 6, final value theorem for generalized Fourier-Laplace transform is established. Lastly conclusions are given in section 7. Notations and terminology as per Zemanian [5],[6].

2 TESTING FUNCTION SPACES

2.1 THE SPACE $FL_{a,\alpha}^{\beta}$

This space is given by

$$FL_{a,\alpha}^{\beta} = \left\{ \phi : \phi \in E_+ \,/\, \rho_{a,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C A^k k^{k\alpha} B^l l^{l\beta} \right\} \tag{2.1}$$

Where, $k,l,q = 0,1,2,3,..........$, and the constants A, B depends on the testing function ϕ .

2.2 THE SPACE $FL_{a,\gamma}$

It is given by

$$FL_{a,\gamma} = \left\{ \phi : \phi \in E_+ \,/\, \xi_{a,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lk} A^q q^{q\gamma} \right\} \tag{2.2}$$

Where, $k,l,q = 0,1,2,3,..........$, and the constants depend on the testing function ϕ .

3 DEFINITIONS

The Fourier transform with parameter s of $f(t)$ denoted by $F\left[f(t)\right] = F(s)$ and is given by

$$F\left[f(t)\right] = F(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)dt \text{, for parameter } s > 0 . \tag{3.1}$$

The Laplace transform with parameter p of $f(x)$ denoted by $L\left[f(x)\right] = F(p)$ and is given by

$$L\left[f(x)\right] = F(p) = \int_0^{\infty} e^{-px} f(x)dx \text{ , for parameter } p > 0 . \tag{3.2}$$

The Conventional Fourier-Laplace transform is defined as

$$FL\left\{f(t,x)\right\} = F(s,p) = \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)K(t,x)dtdx \text{ ,} \tag{3.3}$$

Where, $K(t,x) = e^{-i(st-ipx)}$

4 DISTRIBUTIONAL GENERALIZED FOURIER-LAPLACE TRANSFORMS (FLT)

For $f(t,x) \in FL_{a,\alpha}^{*\beta}$, where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^{\beta}$. It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\{f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle , \tag{4.1}$$

Where, for each fixed t $(0<t<\infty)$, x $(0<x<\infty)$, $s>0$ and $p>0$, the right hand side of (4.1) has a sense as an application of $f(t,x) \in FL^{*\beta}_{a,\alpha}$ to $e^{-i(st-ipx)} \in FL^{\beta}_{a,\alpha}$.

5 Initial Value Theorem for Generalized Fourier-Laplace Transform

Conditions:

(i) $f(t,x)=0$ for $-\infty<t<T$, $0<x<\infty$

(ii) There exist real numbers s and p such that $f(t,x)e^{-i(st-ipx)}$ is absolutely integrable over $-\infty<t<\infty$, $0<x<\infty$.

5.1 Theorem

If the locally integrable function $f(t,x)$ satisfying above conditions with $T=0$ and if there exist a complex number A and a real number m and n such that

(i) $m>-1$ (ii) $n>-1$ and

$$\lim_{t,x\to 0^+} \frac{\Gamma(m+1)}{ct^m x^n} f(t,x) = A \tag{5.1.1}$$

Then: $\lim_{\substack{s\to\infty \\ p\to 0^+}} (is)^{m+1} p^{n+1} F(s,p) = A$, where $c = \dfrac{1}{\Gamma(n+1)}$

Proof: By Zemanian [5] pp.243 and Debnath Lokenath [2] pp. 136 , we know that

(i) $\int_{-\infty}^{\infty} t^m e^{-ist} dt = \dfrac{\Gamma(m+1)}{(is)^{m+1}}$, for $m>-1$ and $s>0$ and

(ii) $\int_{0}^{\infty} x^n e^{-px} dx = \dfrac{\Gamma(n+1)}{p^{n+1}}$, for $n>-1$, $p>0$ so that

$\left|(is)^{m+1} p^{n+1} F(s,p) - A\right|$

$$= (is)^{m+1} p^{n+1} \int_{-\infty}^{\infty}\int_{0}^{\infty} \left| f(t,x) - \frac{cAt^m x^n}{\Gamma(m+1)} \right| e^{-ist} e^{-px} dtdx$$

$$\le (is)^{m+1} p^{n+1} \{ \int_{-\infty}^{T}\int_{0}^{X} \left| f(t,x) - \frac{cAt^m x^n}{\Gamma(m+1)} \right| e^{-i(st-ipx)} dtdx$$

$$+ \int_{T}^{\infty}\int_{0}^{X} \left| f(t,x) - \frac{cAt^m x^n}{\Gamma(m+1)} \right| e^{-i(st-ipx)} dtdx$$

$$+ \int_{-\infty}^{T}\int_{X}^{\infty} \left| f(t,x) - \frac{cAt^m x^n}{\Gamma(m+1)} \right| e^{-i(st-ipx)} dtdx$$

$$+ \int_{T}^{\infty}\int_{X}^{\infty} \left| f(t,x) - \frac{cAt^m x^n}{\Gamma(m+1)} \right| e^{-i(st-ipx)} dtdx \}$$

By Zemanian [5] pp. 243 and D. V. Widder [11] pp.181 for any positive ε we can find a constant M such that

$$(is)^{m+1}\int_T^{\infty} e^{-ist}\left|\alpha(t)-\frac{At^m}{\Gamma(m+1)}\right|dt<\frac{M(is)^{m+1}}{i(s-\varepsilon)}e^{-i(s-\varepsilon)T} \text{ for } s>\varepsilon$$

Where right hand side of this inequality is approaches zero as s becomes infinite and by D. V. Widder [11] pp.181 we have

$$p^{n+1}\int_X^{\infty} e^{-px}\left|\alpha(x)-\frac{Ax^n}{\Gamma(n+1)}\right|dx<\frac{Mp^{n+1}}{p-\varepsilon}e^{-(p-\varepsilon)X} \text{ for } p>\varepsilon \text{, therefore}$$

$$\lim_{\substack{s\to\infty\\ p\to 0^+}}\left|(is)^{m+1}p^{n+1}F(s,p)-A\right|$$

$$\leq \lim_{\substack{s\to\infty\\ p\to 0^+}}(is)^{m+1}p^{n+1}\left\{\int_{-\infty}^{T}\int_0^X \frac{ct^m x^n}{\Gamma(m+1)}\left|\frac{f(t,x)\Gamma(m+1)}{ct^m x^n}-A\right|e^{-i(st-ipx)}dtdx\right\}$$

$$\leq \sup_{\substack{-\infty\leq t\leq T\\ 0\leq x\leq X}}\left|\frac{f(t,x)\Gamma(m+1)}{ct^m x^n}-A\right|$$

Since T and X are arbitrary

$$\leq \lim_{\substack{t\to 0^+\\ x\to 0^+}}\left|\frac{f(t,x)\Gamma(m+1)}{ct^m x^n}-A\right|$$

From which the result follows.

5.2 Lemma

If $f(t,x)\in FL_{a,\alpha}^{\beta}$ with its support in $t_f\leq t\leq\infty$ and $x_f\leq x\leq\infty$ where $t_f>0$ and $x_f>0$ then $\left|F(s,p)\right|\leq Mt^k e^{ax}$, where M is sufficiently large constant.

Proof:

Let $g(t,x)$ be a smooth function on $0\leq t\leq\infty$ and $0\leq x\leq\infty$ such that $g(t,x)=1$ on $\left[t_f,\infty\right)$ and $\left[x_f,\infty\right)$ and $g(t,x)=0$ on $\left(0,T\right)$ and $\left(0,X\right)$ where $T<t_f$ and $X<x_f$. As a distribution of slow growth satisfies a boundedness property of distribution, there exists a positive constant K and a non-negative integer ρ such that

$$\left|F(s,p)\right|\leq K\max_{0\leq t\leq\rho}\sup_{0\leq t,x<\infty}\left|t^k e^{ax}D_t^l D_x^q g(t,x)f(t,x)\right|$$

$$\leq K\max\sup\left|t^k e^{ax}g^{l+q}(t,x)f(t,x)\right|$$

Where, $g^{l+q}(t,x)$ gives l^{th} derivatives of $g(t,x)$ with respect to 't' and q^{th} derivative of $g(t,x)$ with respect to 'x'.

$$\leq K\max\sup t^{k-l}e^{ax}$$

As $f(t,x)$ is a member of $FL^{\beta}_{a,\alpha}$ and $k,l = 0,1,2,..........$

$$\leq K \max\sup t^k e^{ax}$$

$$\leq Mt^k e^{ax}$$

Where M is sufficiently large constant.

5.3 THEOREM

If $f(t,x)$ is decomposed into $f(t,x) = f_1(t,x) + f_2(t,x)$ where $f_1(t,x)$ is the ordinary FL transformable function satisfying the hypothesis of theorem 5.1 and $f_2(t,x)$ is a regular distribution satisfying the hypothesis of above lemma, then

$$\lim_{\substack{s \to \infty \\ p \to 0^+}} (is)^{m+1} p^{n+1} F(s,p) = A$$

PROOF:

It follows from above lemma that

(i) $$\lim_{\substack{s \to \infty \\ p \to 0^+}} (is)^{m+1} p^{n+1} F(s,p) = 0$$

And from theorem 5.1, we have

(ii) $$\lim_{\substack{s \to \infty \\ p \to 0^+}} (is)^{m+1} p^{n+1} F(s,p) = A$$

Moreover the distributional generalized FL transform F_1 of f_1 equals the ordinary generalized FL transform of f_1, so that $F(s,p) = F_1(s,p) + F_2(s,p)$, therefore (i) and (ii) proves theorem.

6 SOME ABELIAN THEOREMS OF THE FINAL VALUE TYPE

6.1 THEOREM

For a locally integrable function $f(t,x)$ satisfying conditions with $T = 0$ and existence of any complex constant A and a real number m and n such that (i) $m > -1$, (ii) $n > -1$ and $\lim_{t,x \to \infty} \frac{\Gamma(m+1) f(t,x)}{ct^m x^n} = A$, then

$$\lim_{\substack{s \to 0^+ \\ p \to \infty}} (is)^{m+1} p^{n+1} F(s,p) = A, \text{ where } c = \frac{1}{\Gamma(n+1)}$$

PROOF:

We proceed as in theorem 5.1 to obtain

$$\left|(is)^{m+1}p^{n+1}F(s,p)-A\right|$$

$$\leq (is)^{m+1}p^{n+1}\{\int_{-\infty}^{T}\int_{0}^{X}\left|f(t,x)-\frac{cAt^m x^n}{\Gamma(m+1)}\right|e^{-i(st-ipx)}dtdx$$

$$+\int_{T}^{\infty}\int_{0}^{X}\left|f(t,x)-\frac{cAt^m x^n}{\Gamma(m+1)}\right|e^{-i(st-ipx)}dtdx$$

$$+\int_{-\infty}^{T}\int_{X}^{\infty}\left|f(t,x)-\frac{cAt^m x^n}{\Gamma(m+1)}\right|e^{-i(st-ipx)}dtdx$$

$$+\int_{T}^{\infty}\int_{X}^{\infty}\left|f(t,x)-\frac{cAt^m x^n}{\Gamma(m+1)}\right|e^{-i(st-ipx)}dtdx\}$$

Therefore, $\lim\limits_{\substack{s\to 0^+\\ p\to\infty}}\left|(is)^{m+1}p^{n+1}F(s,p)-A\right|$

$$\leq \lim_{\substack{s\to 0^+\\ p\to\infty}}(is)^{m+1}p^{n+1}\int_{T}^{\infty}\int_{X}^{\infty}\frac{cAt^m x^n}{\Gamma(m+1)}\left|\frac{\Gamma(m+1)}{cAt^m x^n}f(t,x)-A\right|e^{-i(st-ipx)}dtdx$$

$$\leq \sup\left|\frac{\Gamma(m+1)}{cAt^m x^n}f(t,x)-A\right| \text{ Since T and X are arbitrary.}$$

$$\leq \lim_{t,x\to\infty}\left|\frac{\Gamma(m+1)}{cAt^m x^n}f(t,x)-A\right|$$

Now the proof can be easily completed.

7 CONCLUSION

The Fourier-Laplace Transform is developed and generalized in the distributional sense in this paper. Some testing function spaces are given. This paper is mainly focuses on Abelian theorems of Initial value type and Final value type.

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